Cell-centered Genuinely Multidimensional Upwind Algorithms and Structured Meshes

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A family of cell-centered genuinely multidimensional upwind schemes for structured meshes is developed. Two different approaches for the numerical flux formulation are applied, based on respectively characteristic variable extrapolation and flux extrapolation. The latter is developed within the context of the unified central/dissipation formulation of classical central and upwind schemes. The numerical flux is based on a 4-wave decomposition model of the 2D Euler equations using two arbitrary characteristic directions. A general theoretical analysis of linear convection schemes is the framework for the development of first- and second-order multidimensional upwind schemes. A monotone second-order zero cross-diffusion algorithm is developed introducing classical limiters with multidimensional ratios assuring the monotonicity property. Several combinations of algorithms with different characteristic directions are tested using a multigrid solver. Results near discontinuities are showing a sharper resolution than grid aligned methods. A significant improvement is obtained with the central/dissipation approach concerning robustness and flexibility for implementation of multidimensional methods in standard codes.

1. Introduction

Algorithms with a genuinely multidimensional upwind approach for solving the Euler/Navier-Stokes equations form the subject in this paper. This new approach aims at reducing the mesh dependency introduced by classical schemes that are based on the dimensional splitting approach.

A family of 2D multidimensional upwind schemes have been developed in the past using a cell-centered finite volume approach on structured meshes Hirsch & Lacor [4], Lacor & Hirsch [7]. The basic theoretical framework developed by Hirsch [5] is the optimal diagonalization of the Euler equations into 4 wave equations using a specific choice of two characteristic directions depending on the local flow gradients. Hirsch & Lacor [4] developed a conservative scheme with the numerical flux evaluated using characteristic MUSCL extrapolation,

i.e. variable extrapolation along the characteristic propagation directions. This leads to a family of first and second-order accurate schemes with an improved accuracy compared to classical methods. Since the higher-order characteristic MUSCL was too expensive, alternative second-order more compact schemes have been looked for by VAN RANSBEECK [13].

A general theoretical study of 2D linear compact convection algorithms is developed by HIRSCH [3]. Based on the truncation error expansion of a general 9-point molecule, conditions concerning accuracy, monotonicity and cross-diffusion are determined for families of 6- and 4-point molecules.

In comparison with the characteristic variable extrapolation, a new numerical flux formulation based on multidimensional flux extrapolation is developed. The resulting numerical flux is written as a central flux with additional dissipation terms. It forms a unified approach for the classical central and upwind schemes including TVD high resolution schemes, HIRSCH [2], SWANSON & TURKEL [11]. In the standard methods, the multidimensional problem is solved by a superposition of 1D dissipation models based on differences along the mesh lines. The genuinely multidimensional schemes to be discussed have additional dissipation terms containing mixed differences representing the multidimensional character.

A unique zero cross-diffusion scheme is developed from the theoretical convection analysis, being second-order accurate for the homogeneous convection equation. Based on similar ideas as for classical second-order TVD high resolution schemes, the second-order zero cross-diffusion scheme is made monotone by rewriting it as a correction to a 2D monotone first-order scheme, modified by introducing non-linear limiters to ensure the monotonicity property.

The extension to the 2D Euler equations is based on a generalized form of the 4 wave decomposition method by introducing two arbitrary characteristic directions. This allows the investigation of different directions for optimization of the multidimensional schemes.

Results are obtained with the CWI adaptive multigrid solver described in VAN DER MAAREL, HEMKER & EVERAARS [12]. The multigrid structure is combined with a simplified implicit operator based on a conservative linearization of the standard first-order upwind Flux Difference Splitting scheme. In comparison with classical methods, inviscid results will be shown for a subsonic and supersonic test case on fully refined meshes.

2. 2D UPWIND CONVECTION ALGORITHMS

2.1. Linear convection analysis

The main elements of a general theoretical analysis of the two-dimensional upwind schemes for the linear non-homogeneous convection equation, HIRSCH [3] are repeated below. It is derived for compact molecules not exceeding nine points. The analysis relies on the properties of the truncation error expansion, taking as starting point the computational nine-point stencil with arbitrary coefficients. Writing out the Taylor expansion for the convection equation,

some of the coefficients are specified as to satisfy the consistency conditions. General conditions are defined for properties such as second-order accuracy, monotonicity, cross-diffusion and relations between some of these conditions are determined. For the non-homogeneous equation the condition of vanishing cross-diffusion is less severe than that of second-order accuracy. For the homogeneous equation the condition of zero cross-diffusion is equivalent with the condition of second-order accuracy. The study introduces several families of schemes based on 6-point and 4-point molecules.

A finite volume formulation of the convection algorithm is considered, within a cell-centered approach for structured meshes. Considering the same mesh spacing equal to 1 in both directions, the discretized convection term in cell (i, j) is written as

$$au_x + bu_y \equiv a(u_{i+1/2,j} - u_{i-1/2,j}) + b(u_{i,j+1/2} - u_{i,j-1/2}) \tag{1}$$

where the approximations selected for the cell face values $u_{i\pm1/2,j}$ and $u_{i,j\pm1/2}$ characterize the schemes. The notation used in (1) is clarified by Figure 1.

FIGURE 1. Cell-centered finite volume formulation.

Defining upwind convection schemes for $a, b > 0$, we can focus the analysis to smaller molecules of respectively 6 and 4 points in the upwind direction, with respect to point 0. In the former case, points 0, 1, 2, 3, 4 and 5 will be involved, while in the latter case, only points 0, 1, 2, 3 contribute to the scheme. In order to introduce multidimensional couplings in the upwind schemes, the cell face values should therefore be made dependent on several nodal values. If $u_{i-1/2,j}$ is made dependent on the nodal values at points 1, 2 and 3, the resulting scheme will involve 6 points. Hence, for 4-point molecules, $u_{i-1/2,j}$ should depend only on the nodal values at points 1 and 3. In the following the family of 4-point molecules will be considered.

2.2. Upwind schemes for 4-point molecules

The 4-point molecules, for $a, b > 0$, are defined by the following extrapolation formulae, illustrated in Figure 2a,

$$\begin{aligned} u_{i-1/2,j} &= u_1 - \alpha(u_1 - u_3) \\ u_{i,j-1/2} &= u_2 - \delta(u_2 - u_3) \end{aligned} \tag{2}$$

FIGURE 2. The 4-point molecules for linear convection schemes.

Writing out (1) with the use of (2) and representing the combination $(a\alpha + b\delta)$ by a parameter A, the 4-point convection scheme is uniquely defined in Figure 2b. It is important to observe, that we have actually a *one-parameter* family of schemes, although the parameters α and δ can be chosen independently. Several interesting schemes are recovered by choosing a particular value of A as shown by HIRSCH [3]:

$$\begin{aligned} &\text{monotone 1st order} && : 0 \le A \le \min(a, b) \\ &\text{'2nd order' zero cross diffusion} && : \quad A = \frac{a+b}{2} \end{aligned} \tag{3}$$

Concerning the monotone first-order scheme the lower limit corresponds to the first-order classical upwind scheme that has maximum cross diffusion. The upper limit represents the minimum cross diffusion scheme similar to the scheme of RICE & SCHNIPKE [8]. This scheme has been applied in e.g. LACOR & HIRSCH [7] and showed indeed a marked improvement in shock resolution, compared to standard first-order flux splitting methods. The unique non-monotone zero cross-diffusion scheme, can be considered as the only second-order compact 4-point scheme in space for the homogeneous convection equation. The monotone minimum cross diffusion scheme and the non-monotone zero cross diffusion scheme have also been investigated by SIDILKOVER [9] under the names of respectively N-scheme and 2D scheme.

Considering the case $a, b > 0$ we can reformulate the general 4-point molecule

from Figure 2b by splitting the central part and dissipation term. Introducing difference operators in e.g. the x-direction by

$$\bar{\delta}_x u_{i,j} = u_{i+1,j} - u_{i-1,j},\ \delta_x^+ u_{i,j} = u_{i+1,j} - u_{i,j},\ \delta_x^- u_{i,j} = u_{i,j} - u_{i-1,j} \quad (4)$$

the residual is rewritten,

$$\left(a\bar{\delta}_x + b\bar{\delta}_y - \tfrac{1}{2}(a\delta_x^-\delta_x^+ + b\delta_y^-\delta_y^+ + 2A\delta_x^-\delta_y^-)\right) u_{i,j} \quad (5)$$

The dissipation term in (5) consists of classical 2nd difference terms along the mesh lines coming from the classical first-order upwind scheme and an additional mixed second difference representing the multidimensional upwind character.

3. Dissipation of multidimensional convection schemes

The family of linear 4-point convection schemes described in section 2.2 for positive convection speeds is reformulated for all combinations of signs of a and b. The numerical flux at e.g. $i+1/2, j$ is defined as

$$f^\star_{i+\frac{1}{2},j} = a^+\left(\frac{b^+}{b}u^{++}_{i+\frac{1}{2},j} + \frac{b^-}{b}u^{+-}_{i+\frac{1}{2},j}\right) + a^-\left(\frac{b^+}{b}u^{-+}_{i+\frac{1}{2},j} + \frac{b^-}{b}u^{--}_{i+\frac{1}{2},j}\right) \quad (6)$$

where the superscripts of the interface states represent the signs of the corresponding components a, b. The interface states are determined using the generalized form of the extrapolation formulae (2). Introducing the splitting of central and dissipation part of the numerical flux,

$$f^\star_{i+1/2,j} = \tfrac{1}{2}a(u_{i,j} + u_{i+1,j}) - d_{i+1/2,j} \quad (7)$$

$$d_{i+1/2,j} = \tfrac{1}{2}\beta_{i+1/2,j}\delta u_{i+1/2,j} \quad (8)$$

the numerical dissipation is given by

$$\begin{aligned} d_{i+1/2,j} = \tfrac{1}{2}|a|\delta u_{i+1/2,j} \quad & +\alpha\left[a^+\left(\frac{b^+}{b}\delta u_{i,j-1/2} - \frac{b^-}{b}\delta u_{i,j+1/2}\right)\right. \\ & \left. +a^-\left(\frac{b^+}{b}\delta u_{i+1,j-1/2} - \frac{b^-}{b}\delta u_{i+1,j+1/2}\right)\right] \end{aligned} \quad (9)$$

with $\delta u_{i+1/2,j} = u_{i+1,j} - u_{i,j}$ and $\delta u_{i,j+1/2} = u_{i,j+1} - u_{i,j}$. Introducing 2D ratios the dissipation (9) can be rewritten in the form of (8). In contrast with a 1D definition of R, built up by 2 consecutive flux differences taken along the same mesh line, one can define 2D ratios based on flux differences in the two mesh directions. This 2D definition introduces the dependence on the sign of both velocity components a and b of the convection speed, where the ratio is defined in a 2D upwind manner relative to the direction of convection speed. Different ratios have been investigated by Sidilkover [9], Hirsch & Van

RANSBEECK [6]. In contradiction with the former, the latter introduced a new 2D ratio which is related to the definition of the dissipation formulation (8). The 2D ratio is defined such that a compact formulation is recovered which is not the case in the 1D formulation. As a result, the six surrounding cell-centers of the wall are considered for the determination of the ratio. Four ratios are defined for every wall, depending each on one of the 4 combinations of the signs of a and b, as shown in Figure 3,

FIGURE 3. 2D definition of ratios.

where the superscripts of R represent the signs of the corresponding components a, b. The corresponding formulae are given by,

$$\begin{aligned} R^{++}_{i+1/2,j} &= \frac{b^+ \delta u_{i,j-1/2}}{a^+ \delta u_{i+1/2,j}}, \qquad R^{+-}_{i+1/2,j} = -\frac{b^- \delta u_{i,j+1/2}}{a^+ \delta u_{i+1/2,j}} \\ R^{-+}_{i+1/2,j} &= -\frac{b^+ \delta u_{i+1,j-1/2}}{a^- \delta u_{i+1/2,j}}, \quad R^{--}_{i+1/2,j} = \frac{b^- \delta u_{i+1,j+1/2}}{a^- \delta u_{i+1/2,j}} \end{aligned} \tag{10}$$

The resulting dissipation coefficient β based on flux extrapolation is determined from (8)-(10),

$$\begin{aligned} \beta_{i+1/2,j} &= |a| - 2\alpha \frac{|a|}{|b|} \Phi(R_{i+1/2,j}) \\ \Phi(R) &= a^+(R^{+-} - R^{++}) + a^-(R^{--} - R^{-+}) \end{aligned} \tag{11}$$

In the non-linear case a Roe type linearisation is introduced in (10)-(11) by e.g.

$$\delta f^{\pm}_{i+1/2,j} = a^{\pm}_{i+1/2,j} \delta u_{i+1/2,j}, \; \delta g^{\pm}_{i,j+1/2} = b^{\pm}_{i,j+1/2} \delta u_{i,j+1/2} \tag{12}$$

The value of the coefficient α in (11) corresponds to the interpolation coefficient in (2) and is determined by the parameter A in (3) that represents a specific multidimensional convection scheme. The interpolation coefficients α and δ for

the monotone minimum cross diffusion scheme and second-order non-monotone zero cross diffusion scheme are chosen by

$$\begin{aligned} \alpha_{mo} &= \frac{1}{2}\min\left(1, \frac{|b|}{|a|}\right), \quad & \delta_{mo} &= \frac{1}{2}\min\left(1, \frac{|a|}{|b|}\right) \\ \alpha_{nmo} &= \frac{1}{2}\frac{|b|}{|a|}, & \delta_{nmo} &= \frac{1}{2}\frac{|a|}{|b|} \end{aligned} \tag{13}$$

4. Monotone second-order zero cross diffusion scheme

4.1. High resolution and monotonicity

The compact scheme with zero cross diffusion introduced in Sections 2.2 and 3 will be more accurate than its first-order monotone counterparts. However, being non-monotone, it will generate oscillations at discontinuities. The technique for avoiding this situation and for generating high resolution schemes without numerical oscillations is well developed for classical second-order TVD high resolution schemes, Hirsch [2]. The basic ideas can be extended for multidimensional schemes. They consist of: i) selecting a first-order monotone numerical flux; ii) extending the numerical flux to second-order accuracy; iii) restricting the amplitude of the gradients appearing in the additional term via non-linear multidimensional limiters, such as to assure the monotonicity property.

Based on the extrapolation formulae (2), the non-monotone zero cross diffusion scheme can be written as a correction to a first-order multidimensional monotone scheme where the additional term is limited in a non-linear way to assure a monotone zero cross diffusion scheme. Consider the linear case $a, b > 0$,

$$\begin{aligned} f^{*zc}_{i+1/2,j} &= f^{*mo}_{i+1/2,j} - a(\alpha_{nmo} - \alpha_{mo})\psi_{i+1/2,j}\delta u_{i,j-1/2} \\ g^{*zc}_{i,j+1/2} &= g^{*mo}_{i,j+1/2} - b(\delta_{nmo} - \delta_{mo})\psi_{i,j+1/2}\delta u_{i-1/2,j} \end{aligned} \tag{14}$$

with

$$\begin{aligned} f^{*mo}_{i+1/2,j} &= au_{ij} - a\alpha_{mo}\delta u_{i,j-1/2} \\ g^{*mo}_{i,j+1/2} &= bu_{i,j} - b\delta_{mo}\delta u_{i-1/2,j} \end{aligned} \tag{15}$$

where α_{mo}, α_{nmo} and δ_{mo}, δ_{nmo} are determined by e.g. (13), and ψ represents a limiter.

For one-dimensional flows, the concept of TVD, Total Variation Diminishing, has been developed (see for instance Hirsch [2]), leading to the introduction of non-linear limiters. Unfortunately, this concept does not appear to be well adapted for multidimensional flows, as shown by Goodman & Leveque [1], although the ad-hoc application of the one-dimensional TVD approach, coupled to directional splitting of the convective fluxes, gives excellent results in practical codes. It is therefore appropriate, for multidimensional configurations, to apply alternative criteria.

A straightforward definition of monotonicity to be considered in a multidimensional framework can be obtained by the approach of SPEKREIJSE [10], which appears as best suited in the present context. Based on the condition that the sum of the coefficients of the 9-point molecule in Figure 1 has to be zero, the schemes are monotone or positive if all coefficients except at point 0 are of the same sign. This is the definition applied by SPEKREIJSE [10], where it is shown that a 1D monotone scheme is TVD, while this is not necessarily the case in 2D. The important consequence of the monotonicity condition is that the steady state solution at (i, j) is a bounded average of the surrounding mesh point values. Therefore, no over- or undershoots can appear in the solution.

4.2. Multidimensional limiters

The limiter function ψ in equation (14) is acting on a 2D ratio. Two definitions of ratios for $a, b > 0$ are introduced, e.g. at wall $i - 1/2, j$ in Figure 4,

FIGURE 4. Definition of multidimensional ratios.

where r^S represents the ratio investigated by SIDILKOVER [9] and r^D is the ratio developed by HIRSCH & VAN RANSBEECK [6]. They are formulated by

$$\begin{aligned} &\psi_{i-1/2,j} = \psi(r_{i-1/2,j}) \quad \text{with} \\ &r^S_{i-1/2,j} = \frac{a(u_3 - u_2)}{b(u_1 - u_3)}, \quad r^D_{i-1/2,j} = \frac{a(u_0 - u_1)}{b(u_1 - u_3)} \end{aligned} \tag{16}$$

In the context of the central/dissipation splitting the definition of r^D is most suitable and is used in the present paper. The superscript D has been dropped for clarity. In order to achieve a monotone second-order zero cross diffusion scheme (14), HIRSCH & VAN RANSBEECK [6] showed that the *multidimensional* limiter ψ is equal to a classical limiter based on the multidimensional ratio considered, that fulfills

$$0 \leq \psi(r) \leq 2, \quad 0 \leq \frac{\psi(r)}{r} \leq 2 \tag{17}$$

4.3. Equivalent dissipation terms

Introducing the central/dissipation approach (7)-(8) in the numerical flux of the monotone zero cross diffusion scheme (14) for the linear case $a, b > 0$, the dissipation is given by

$$d^{ZC}_{i+1/2,j} = \tfrac{1}{2}|a|\delta u_{i+1/2,j} + a^{+}(\alpha_{mo} + (\alpha_{nmo} - \alpha_{mo})\psi(r^{++}_{i+1/2,j}))\delta u_{i,j-1/2} \tag{18}$$

A new ratio R is derived and the symmetry property of the classical limiters is introduced,

$$R = \frac{1}{r}, \quad \psi(R) = \psi(\frac{1}{r}) = \frac{\psi(r)}{r} \tag{19}$$

where R corresponds to the definition in Figure 3 and (10). As a result the dissipation term (18) is written as,

$$\begin{aligned} d^{ZC}_{i+1/2,j} \quad &= \frac{1}{2}|a|\delta u_{i+1/2,j} \\ &+a^{+}\frac{|a|}{|b|}\left(\alpha_{mo}R^{++}_{i+1/2,j} + (\alpha_{nmo} - \alpha_{mo})\psi(R^{++}_{i+\frac{1}{2},j})\right)\delta u_{i+1/2,j} \end{aligned} \tag{20}$$

Considering the four possible combinations of signs of a and b, a general formulation of the dissipation coefficient is derived

$$\beta^{ZC}_{i+1/2,j} = |a| - 2\frac{|a|}{|b|}\left(\alpha_{mo}\Phi(R_{i+1/2,j}) + (\alpha_{nmo} - \alpha_{mo})\Phi(\psi(R_{i+1/2,j}))\right) \tag{21}$$

where the function Φ is determined by (11).

5. Decomposition of the Euler equations

The theoretical diagonalisation based on characteristic theory, Hirsch [5] is reformulated in a general way by considering the arbitrary choice of two characteristic directions. The quasi-linear formulation of the 2D Euler equations can be derived from the conservative differential form,

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0 \tag{22}$$

based on the conservative variables U, with F, G representing respectively the flux vectors in the x - and y - direction with corresponding Jacobians A and B. A similarity transformation based on two arbitrary characteristic directions κ_1, κ_2, is defined by

$$P^{*-1}(\kappa_1, \kappa_2)\vec{A}P^{*}(\kappa_1, \kappa_2) = \vec{\Lambda}(\kappa_1, \kappa_2) + \vec{C}(\kappa_1, \kappa_2) \tag{23}$$

where P^* is the transformation matrix with columns representing the right eigenvectors of the Jacobian. The diagonal matrix $\vec{\Lambda}$ and the non-diagonal matrix $\vec{C}$ represent respectively the characteristic propagation speeds of the diagonalisation and the source or coupling matrix. A set of characteristic variables can be constructed such that

$$\partial W = P^{*-1} \partial U \tag{24}$$

Using equations (23) - (24), the Euler equations (22) are recast into the following set of characteristic compatibility equations

$$\frac{\partial W}{\partial t} + (\vec{\Lambda}.\vec{\nabla})W + (\vec{C}.\vec{\nabla})W = 0 \tag{25}$$

The corresponding expressions of Jacobians, eigenvectors, eigenvalues and characteristic variables are presented explicitly in HIRSCH, LACOR & DECONINCK [5]. They show that the source terms in (25) are identical to zero if the two characteristic directions are chosen such that

$$\vec{\kappa}_1 \times \vec{\nabla} p = 0, \ \vec{\kappa}_2.(\vec{\kappa}_2.\vec{\nabla})\vec{v} - \vec{\nabla}.\vec{v} = 0 \tag{26}$$

The first characteristic direction is chosen along the pressure gradient and the second one is related to the strain rate tensor. Taking the characteristic directions along the gridnormals results in the standard diagonalisation of the linear combination of the Jacobians with the gridnormal, YEE [14],

$$P^{*-1}(\vec{A}.\vec{n})P^* = \vec{\Lambda}.\vec{n} \tag{27}$$

6. NUMERICAL FLUX FORMULATION

6.1. Characteristic MUSCL extrapolation

The characteristic system of non-linear scalar convection equations (25) with possible source terms is discretized using a cell-centered finite volume approach for structured meshes. An upwind discretization is used for the convective part and a central scheme for the possible coupling terms. The general formulation of the numerical flux based on the characteristic MUSCL extrapolation derived by HIRSCH & LACOR [4], is written as

$$\begin{aligned} (\vec{F}.\vec{n})^*_{i+1/2,j} = \quad & \tfrac{1}{2}(\vec{F}^+_{i+1/2,j} + \vec{F}^-_{i+1/2,j}).\vec{n}_{i+1/2,j} \\ & -\tfrac{1}{2} P^*_{i+1/2,j} \left| \vec{\Lambda}.\vec{n} \right|_{i+1/2,j} (W^-_{i+1/2,j} - W^+_{i+1/2,j}) \end{aligned} \tag{28}$$

where $W^+_{i+1/2,j}$ and $W^-_{i+1/2,j}$ are the interface values with the superscripts $+$ and $-$ indicating upwind and downwind extrapolations with respect to the direction of the outward pointing unit normal $\vec{n}$ on the cell face. Notice that the characteristic variables W, the eigenvectors P^* and eigenvalues $\vec{\Lambda}.\vec{n}$ are

based on Roe averages and two arbitrary characteristic directions κ_1, κ_2 on the appropriate wall. The first term between brackets in the right-hand-side of (28) is calculated using

$$\begin{aligned}
&(\vec{F}^{\pm}.\vec{n})_{i+1/2,j} = \vec{F}.\vec{n}(U^{\pm}_{i+1/2,j}) \\
&U^{+}_{i+1/2,j} = U_{i,j} + P^{*}(W^{+}_{i+1/2,j} - W_{i,j}) \\
&U^{-}_{i+1/2,j} = U_{i+1,j} + P^{*}(W^{-}_{i+1/2,j} - W_{i+1,j})
\end{aligned} \tag{29}$$

that is based on (24). The variations of the characteristic variables in (28) - (29) are obtained using one of the compact multidimensional convection algorithms from Section 2 under the form of variable extrapolation. The determination of left and right state corresponding to a chosen algorithm depends on the characteristic propagation directions,

$$\vec{\lambda}^{(1)} = \vec{\lambda}^{(2)} = \vec{v},\ \vec{\lambda}^{(3)} = \vec{v} + c\vec{\kappa_2},\ \vec{\lambda}^{(4)} = \vec{v} - c\vec{\kappa_2} \tag{30}$$

with $\vec{\lambda}^{(k)}$ the propagation direction corresponding to component w^k of W.

The classical first-order upwind scheme based on flux difference splitting is recovered from (28) by considering the grid normals as the two characteristic directions (27) and the upwind and downwind states as the values in the neighbouring cells of the wall,

$$(\vec{F}.\vec{n})^{*}_{i+1/2,j} = \tfrac{1}{2}(\vec{F}_{i,j} + \vec{F}_{i+1,j}).\vec{n}_{i+1/2,j} - \tfrac{1}{2}\left(|\vec{A}.\vec{n}|\delta U\right)_{i+1/2,j} \tag{31}$$

6.2. Flux extrapolation - central/dissipation formulation

Concerning the new approach based on multidimensional flux extrapolation, the convective terms of (25) are discretized using the central/dissipation formulation (7)-(8) discussed in Sections 3 and 4 for the 2D convection schemes of section 2.2. Consider the numerical flux in the x-direction for the convective terms in (25) on a Cartesian mesh,

$$\begin{aligned}
&W^{*}_{i+1/2,j} = \tfrac{1}{2}\Lambda_x(W_{i,j} + W_{i+1,j}) - D_{i+1/2,j} \\
&D_{i+1/2,j} = \tfrac{1}{2}\mathrm{diag}(\beta^{k}_{i+1/2,j})\delta W_{i+1/2,j}
\end{aligned} \tag{32}$$

where D represents the numerical dissipation matrix and $\mathrm{diag}(\beta^k)$ is the diagonal matrix consisting of the numerical dissipation coefficient β^k of each of the 4 characteristic equations. Defining a Roe type linearization the total numerical flux on an irregular mesh for e.g. wall i+1/2,j, is derived by Hirsch & Van Ransbeeck [6],

$$(\vec{F}.\vec{n})^{*}_{i+1/2,j} = \tfrac{1}{2}(\vec{F}_{i,j} + \vec{F}_{i+1,j}).\vec{n}_{i+1/2,j} - \tfrac{1}{2}(P^{*}\mathrm{diag}(\beta^{k})\delta W)_{i+1/2,j} \tag{33}$$

The dissipation coefficient β^k is based on the convection speed $\vec{\lambda}^{(k)}$ of (30), and corresponds to the non-linear form of (11) or (21) for a specific first or

second-order multidimensional convection scheme.

7. Solution method

The solution method for the discretized Euler equations (22) uses multigrid acceleration. As smoothing procedure a Collective Symmetric Gauss-Seidel (CSGS-) relaxation is applied. The 4 non-linear equations are solved by Newton's method (local linearisation). Rewriting the non-linear system of discretized Euler equations for a cell i,j on level k, yields,

$$(\sum_{\text{sides}}(\vec{F}.\vec{n})^*\Delta S)^k_{i,j} = (N^k(U^k))_{i,j} = (r^k)_{i,j} \tag{34}$$

where ΔS is the cell face length, r represents the right hand side and $N(U)$ the residual. In a cell not on the finest level the right-hand side is

$$r^k_{i,j} = (N^k(U^k))_{i,j} - (I^k_{k+1}(N^{k+1}(U^{k+1}) - r^{k+1}))_{i,j} \tag{35}$$

where I^k_{k+1} is the restriction operator working on the defect $N^{k+1}(U^{k+1}) - r^{k+1}$. In a cell on the finest level the right-hand side is

$$r^k_{i,j} = 0 \tag{36}$$

One Newton iteration applied to (34) using (35) and (36) for cell i,j is defined by

$$\begin{aligned} &(r^k)_{i,j} - (N^k U^k)_{i,j} = \frac{\partial (N^k U^k)_{i,j}}{\partial (U^k)_{i,j}}.(\delta U^k)_{i,j} \\ &(U^k)^{\text{new}}_{i,j} = (U^k)^{\text{old}}_{i,j} + (\delta U^k)_{i,j} \end{aligned} \tag{37}$$

The computation of the Jacobian matrix in (37) is based on a conservative linearization of the central part of the first-order numerical flux (31),

$$\left((\vec{F}.\vec{n})^*_{\text{central}}\right)^{n+1}_{i+1/2,j} = \left((\vec{F}.\vec{n})^*_{\text{central}}\right)^{n}_{i+1/2,j} + \left((\vec{A}.\vec{n})^n \Delta U\right)_{i+1/2,j} \tag{38}$$

where

$$\begin{aligned} &(\Delta U)_{i+1/2,j} = \tfrac{1}{2}(\Delta U_{i,j} + \Delta U_{i+1,j}) \\ &\Delta U = U^{n+1} - U^n \end{aligned} \tag{39}$$

Writing out the residual for cell i,j on time step $n+1$, and evaluating the term $\delta U_{i,j}$, one finds

$$\frac{\partial (NU)_{i,j}}{\partial (U)_{i,j}} = (\vec{A}.\vec{n})^+_{i+1/2,j} + (\vec{A}.\vec{n})^+_{i,j+1/2} - (\vec{A}.\vec{n})^-_{i-1/2,j} - (\vec{A}.\vec{n})^-_{i,j-1/2} \tag{40}$$

with

$$(\vec{A}.\vec{n})^{\pm} = \frac{\partial(\vec{F}.\vec{n})^{\pm}}{\partial U} = P^{*}(\vec{\Lambda}.\vec{n})^{\pm}P^{*-1} \tag{41}$$

Equation (41) is similar to (27), except that the eigenvalues have to be replaced by respectively their positive/negative values.

8. Results

All results are obtained using a 3 level V-cycle multigrid acceleration. A shock reflection is investigated in a supersonic rectangular channel $(0,4)\times(0,1)$ using a mesh of $48*16$ cells with an inlet Mach number of 2.9 and a deflection angle of 10.94 degrees. Using the flux extrapolation approach, the monotone minimum cross diffusion and non-monotone zero cross diffusion schemes, referred to as the monotone and non-monotone compact scheme are tested, combined with the characteristic directions along the grid normals, velocity and pressure gradient. The iso-density lines and density distribution along section $y = 0.5$ are shown respectively in Figures 5 and 6. Compared to classical first-order upwind (Flux Difference Splitting) all compact results are showing a better resolution near the shocks. Based on the pressure gradient, the monotone and especially the non-monotone compact scheme show an improvement compared with a classical second-order result. The characteristic directions based on the pressure gradient blended with the velocity are shown in Figure 7. Considering the monotone and non-monotone scheme in combination with pressure gradient, a comparison between the variable extrapolation and the central/dissipation approach shows a negligible difference in accuracy. The main advantage of the central/dissipation approach is the improvement of convergence properties and robustness (Figure 8). The variable extrapolation method requires the freezing of the directions in order to converge to machine accuracy while for the flux extrapolation no freezing is applied and a much better convergence rate is reached. The monotone second-order zero cross diffusion scheme was tested with different classical limiters based on the definition of the multidimensional ratio r^{D} (16). A comparison with the monotone minimum cross diffusion scheme and non-monotone zero cross diffusion scheme in Figure 9 shows that the van Albada limiter gives the sharpest resolution of the shock.

The second test case deals with a fully subsonic flow in a channel $(0,4)\times(0,1)$ with a 20% sinus bump on the lower wall. The inlet Mach number is 0.5 with a mesh of $48*16$ cells. A comparison for the non-monotone zero cross diffusion scheme, with classical first- and second-order results is shown in Figure 10, 11. The non-monotone scheme based on the pressure gradient shows the same accuracy as the second-order classical result, except near the wall. Figure 12 shows the corresponding directions. Based on the flux extrapolation approach an impressive improvement is reached for the convergence to machine accuracy (Figure 13).

9. Conclusions

A family of cell-centered genuinely 2D upwind algorithms based on characteristic variable and flux extrapolation have been developed for structured meshes. The latter formulation introduces the splitting of central and dissipation part. This approach leads to new multidimensional dissipation models as an alternative to the standard 1D models. A monotone second-order zero cross diffusion scheme has been developed, introducing classical limiters based on multidimensional ratios. The flux extrapolation approach shows the advantage of improving significantly the convergence properties and robustness of the schemes. In addition, this formulation has the potential of generality and flexibility for implementation in standard codes.

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FIGURE 5. Shock reflection, iso-density lines.

FIGURE 6. Density distribution along section $y = 0.5$.

FIGURE 7. Characteristic directions $\vec{\bigtriangledown} p/\vec{v}$.

FIGURE 8. Convergence history-RMS density.

FIGURE 9. Investigation multidimensional limiters.

FIGURE 10. Subsonic bump, iso-density lines.

FIGURE 11. Density distribution, section $x = 2.0$.

FIGURE 12. Characteristic direction $\vec{\nabla} p / \vec{v}$.

FIGURE 13. Convergence history-RMS density.